LIBRARY
OF THE
MASSACHUSETTS INSTITUTE
OF TECHNOLOGY

HD28
.M414
no.85-64

WORKING PAPER
ALFRED P. SLOAN SCHOOL OF MANAGEMENT

Multinormal Bayesian Analysis: Two Examples

by

J. J. Martin

MASS. INST. TECH.
OCT 30 1964
DEWEY LIBRARY

May, 1964

85-64

MASSACHUSETTS
INSTITUTE OF TECHNOLOGY
50 MEMORIAL DRIVE
CAMBRIDGE, MASSACHUSETTS 02139

Multinormal Bayesian Analysis: Two Examples

by

J. J. Martin

May, 1964

85-64

MASS. INST. TECH.
OCT 30 1964
DEWEY LIBRARY

This research was done under the direction of G.M. Kaufman, Assistant Professor of Industrial Management, Alfred P. Sloan School of Management, as part of the requirements for the course 15.951, Special Studies in Management, during the second term, 1963-1964.

HD28
M414
no. 85-64

Dewey

MASS. INST. TECH.
AUG 26 1975
LIBRARIES

GENERAL BOOKBINDING CO.
QUALITY CONTROL MARK

Multinormal Bayesian Analysis: Two Examples

J. J. Martin

1. Introduction

The Bayesian analysis of the independent multinormal process in which neither the mean nor precision are known with certainty is carried out by Ando and Kaufman in [1]. These same authors extend this analysis in [2] to a generalized regression model, termed the reduced form data generating process. In order to illustrate these results and to suggest the manner in which multinormal Bayesian analysis can be applied to some vital areas of operations research, two examples are presented here. Particular attention is given, in both cases, to the problem of assigning a prior distribution to the parameters of the process. In section 2 the moments of certain distributions are derived for use in the sequel. In section 3 a radar-tracking model is presented, in which the error in the target velocity has a bivariate normal distribution. The Bayesian analysis of a system of simultaneous linear equations is illustrated in section 4, using Leontief's static inter-industry model. The mean and variance of the standardized and non-standardized inverted generalized Beta distributions can be obtained using the theorems of section 2. These results are presented in Appendix A.

1751891

2. Distribution Theory

The moments of certain distributions will be required in order to fit prior distributions. These moments are derived in this section.

The notation used here follows, in general, that of Raiffa and Schlaifer [3]. In particular, random variables are identified by means of the tilde; e.g., $\tilde{a}$, $\tilde{\underline{\mu}}$, and $\tilde{\underline{\underline{h}}}$ are, respectively, a random variable, a random vector, and a random matrix. The matrix with generic element h_{ij} is denoted by $\underline{\underline{h}}$; a vector with generic element μ_i is written $\underline{\mu}$. All vectors are column vectors unless otherwise noted; column vectors will, on occasion, be written $\{\mu_1,\ldots,\mu_r\}$. The transpose of the matrix $\underline{\underline{h}}$ is $\underline{\underline{h}}^t$. The elements of $\underline{\underline{h}}^{-1}$ are written h_{ij}^{-1}, and are not to be confused with $1/h_{ij}$. The symbol $E_{x|y}[\tilde{z}]$ denotes the expected value of $\tilde{z}$ with respect to the conditional distribution of $\tilde{x}$ given $\tilde{y}=y$; $E_x[\tilde{z}]$ is the expected value of $\tilde{z}$ relative to the unconditional distribution of $\tilde{x}$.

Theorem 1. Let the r x 1 vector $\tilde{\underline{\mu}}$ and the r x r positive definite symmetric (PDS) matrix $\tilde{\underline{\underline{h}}}$ have the Normal-Wishart distribution,

$$f_{NW}^{(r)}(\underline{\mu}, \underline{\underline{h}}|\underline{m}, \underline{\underline{V}}, n, \nu) = f_N^{(r)}(\underline{\mu}|\underline{m},n\underline{\underline{h}}) \quad f_W^{(r)}(\underline{\underline{h}}|\underline{\underline{V}}, \nu) .$$

Let $\tilde{\underline{h}}=\{\tilde{h}_1,\ldots,\tilde{h}_R\}$ be a random vector consisting of the $R=\frac{1}{2}r(r+1)$ distinct elements of $\tilde{\underline{\underline{h}}}$,

$$\tilde{\underline{h}}=\{\tilde{h}_{11},\ldots,\tilde{h}_{1r},\tilde{h}_{22},\ldots,\tilde{h}_{rr}\} \tag{1}$$

and let

$$\bar{\underline{h}}=\{\bar{h}_1,\ldots,\bar{h}_R\} = (\nu+r-1)\ \{V_{11}^{-1},\ldots,V_{1r}^{-1},V_{22}^{-1},\ldots,V_{rr}^{-1}\} . \tag{2}$$

Then

$$E\begin{bmatrix}\tilde{\underline{\mu}}\\ \tilde{\underline{h}}\end{bmatrix} = \begin{bmatrix}\underline{m}\\ \bar{\underline{h}}\end{bmatrix} , \tag{3}$$

and the variance-covariance matrix of $\{\tilde{\underline{\mu}}, \tilde{\underline{h}}\}$ is, provided $\nu > 2$,

$$V\begin{bmatrix}\tilde{\underline{\mu}}\\ \tilde{\underline{\underline{h}}}\end{bmatrix} = \begin{bmatrix}\frac{1}{n(\nu-2)}\underline{\underline{V}} & \underline{\underline{0}}\\ \underline{\underline{0}} & \underline{\underline{U}}\end{bmatrix} \tag{4}$$

where, if $\tilde{h}_i = \tilde{h}_{\alpha\beta}$ and $\tilde{h}_j = \tilde{h}_{\gamma\delta}$,

$$U_{ij} = \mathrm{cov}[\tilde{h}_{\alpha\beta}, \tilde{h}_{\gamma\delta}] = (\nu+r-1)[V^{-1}_{\alpha\gamma}\, V^{-1}_{\beta\delta} + V^{-1}_{\alpha\delta}\, V^{-1}_{\beta\gamma}] \ . \tag{5}$$

If i=j, we obtain

$$U_{ii} = \mathrm{var}[\tilde{h}_{\alpha\beta}] = (\nu+r-1)[(V^{-1}_{\alpha\beta})^2 + V^{-1}_{\alpha\alpha}\, V^{-1}_{\beta\beta}], \tag{6}$$

Proof: Since the marginal distribution of $\tilde{\underline{\mu}}$ is the non-degenerate multivariate Student distribution* with parameters $(\underline{m}, \nu n\underline{\underline{V}}^{-1}, \nu)$,

$$\begin{aligned} E[\tilde{\underline{\mu}}] &= E_{\underline{\mu}} E_{\underline{\underline{h}}|\underline{\mu}}[\tilde{\underline{\mu}}] = \int_{R_{\underline{\mu}}}\int_{R_{\underline{\underline{h}}}} \underline{\mu}\, f_N^{(r)}(\underline{\mu}|\underline{m}, n\underline{\underline{h}})\ \ f_W^{(r)}(\underline{\underline{h}}|\underline{\underline{V}}, \nu)d\underline{\underline{h}}\, d\underline{\mu} \\ &= \int_{R_{\underline{\mu}}} \underline{\mu} f_S^{(r)}(\underline{\mu}|\underline{m}, \nu n\underline{\underline{V}}^{-1}, \nu)d\underline{\mu} = \underline{m} \ . \end{aligned} \tag{7}$$

Similarly, since the marginal distribution of $\tilde{\underline{\underline{h}}}$ is Wishart** with parameters $(\underline{\underline{V}}, \nu)$, it follows that†

$$E[\tilde{h}_{ij}] = E_{\underline{\underline{h}}}[\tilde{h}_{ij}] = (\nu+r-1)V^{-1}_{ij} \ . \tag{8}$$

Equations (2) and (3) follow from (7) and (8).

The variance-covariance matrix of $\{\tilde{\underline{\mu}}, \tilde{\underline{\underline{h}}}\}$ is

$$E\left[\begin{pmatrix}\tilde{\underline{\mu}}-\underline{m}\\ \tilde{\underline{\underline{h}}}-\bar{\underline{\underline{h}}}\end{pmatrix}[(\tilde{\underline{\mu}}-\underline{m})^t, (\tilde{\underline{\underline{h}}}-\bar{\underline{\underline{h}}})^t]\right] = E\begin{bmatrix}(\tilde{\underline{\mu}}-\underline{m})(\tilde{\underline{\mu}}-\underline{m})^t & (\tilde{\underline{\mu}}-\underline{m})(\tilde{\underline{\underline{h}}}-\bar{\underline{\underline{h}}})^t\\ (\tilde{\underline{\underline{h}}}-\bar{\underline{\underline{h}}})(\tilde{\underline{\mu}}-\underline{m})^t & (\tilde{\underline{\underline{h}}}-\bar{\underline{\underline{h}}})(\tilde{\underline{\underline{h}}}-\bar{\underline{\underline{h}}})^t\end{bmatrix} \tag{9}$$

---

*[1], p. 5.

**[1], p. 6.

† [4], p. 161. Note that the Wishart distribution as parameterized here and in [1] and [2] uses (ν+r-1) in place of the parameter n of [4].

But

$$E[(\tilde{\underline{\mu}}-\underline{m})(\tilde{\underline{h}}-\bar{\underline{h}})^t] = E_{\underline{\underline{h}}}\, E_{\underline{\mu}|\underline{\underline{h}}}[(\tilde{\underline{\mu}}-\underline{m})(\tilde{\underline{h}}-\bar{\underline{h}})^t] = \underline{\underline{0}} \quad , \tag{10}$$

$$E[(\tilde{\underline{\mu}}-\underline{m})(\tilde{\underline{\mu}}-\underline{m})^t] = E_{\underline{\mu}}[(\tilde{\underline{\mu}}-\underline{m})(\tilde{\underline{\mu}}-\underline{m})^t] = \frac{1}{n(\nu-2)}\,\underline{\underline{V}} \quad , \quad \nu > 2, \tag{11}$$

and

$$E[(\tilde{\underline{h}}-\bar{\underline{h}})(\tilde{\underline{h}}-\bar{\underline{h}})^t] = E_{\underline{\underline{h}}}[(\tilde{\underline{h}}-\bar{\underline{h}})(\tilde{\underline{h}}-\bar{\underline{h}})^t] = \underline{\underline{U}} \quad . \tag{12}$$

The last equation follows from [4], p. 161. Q.E.D.

<u>Theorem 2.</u> If the r x r PDS matrix $\tilde{\underline{\underline{h}}}$ has the Wishart distribution with parameters $\underline{\underline{V}}$ and ν, then

$$E[\tilde{\underline{\underline{h}}}^{-1}] = \frac{1}{\nu-2}\,\underline{\underline{V}} \quad , \qquad \nu > 2 \ . \tag{13}$$

Let

$$\tilde{\underline{h}}^{-1} = \{\tilde{h}^{-1}_{11},\ldots,\tilde{h}^{-1}_{1r},\tilde{h}^{-1}_{22},\ldots,\tilde{h}^{-1}_{rr}\} \tag{14}$$

and

$$\bar{\underline{h}}^{-1} = E[\tilde{\underline{h}}^{-1}] \quad .$$

Then, provided $\nu > 4$,

$$V[\tilde{\underline{h}}^{-1}] = E[(\tilde{\underline{h}}^{-1}-\bar{\underline{h}}^{-1})(\tilde{\underline{h}}^{-1}-\bar{\underline{h}}^{-1})^t] = \underline{\underline{W}}(\underline{\underline{V}}) \quad , \tag{15}$$

where, if $\tilde{h}^{-1}_i=\tilde{h}^{-1}_{\alpha\beta}$ and $\tilde{h}^{-1}_j=\tilde{h}^{-1}_{\gamma\delta}$,

$$W_{ij}(\underline{\underline{V}}) = \mathrm{cov}[\tilde{h}^{-1}_{\alpha\beta},\ \tilde{h}^{-1}_{\gamma\delta}] = \frac{1}{3(\nu-2)(\nu-4)}\left[\frac{2(5-\nu)}{\nu-2}\, V_{\alpha\beta}V_{\gamma\delta}+V_{\alpha\gamma}V_{\beta\delta}+V_{\alpha\delta}V_{\beta\gamma}\right] . \tag{16}$$

If i=j, we obtain the variance of $\tilde{h}^{-1}_{\alpha\beta}$,

$$W_{ii}(\underline{\underline{V}}) = \mathrm{var}[\tilde{h}^{-1}_{\alpha\beta}] = \frac{1}{3(\nu-2)(\nu-4)}\left[\frac{8-\nu}{\nu-2}\, V_{\alpha\beta}^2 + V_{\alpha\alpha}V_{\beta\beta}\right] \quad . \tag{17}$$

<u>Proof</u>: Let $(\tilde{\underline{\mu}}, \tilde{\underline{\underline{h}}})$ have the Normal-Wishart distribution with parameters $(\underline{0}, \underline{\underline{V}}, 1, \nu)$. Consider, for any real r x 1 vector $\underline{u}$, the characteristic function of the marginal distribution of $\tilde{\underline{\mu}}$,

$$E[e^{i\,\underline{u}^t\tilde{\underline{\mu}}}] = E_{\underline{\mu}}E_{\underline{\underline{h}}|\underline{\mu}}[e^{i\,\underline{u}^t\tilde{\underline{\mu}}}] = E_{\underline{\underline{h}}}E_{\underline{\mu}|\underline{\underline{h}}}[e^{i\,\underline{u}^t\tilde{\underline{\mu}}}] \quad . \tag{18}$$

Expressing $e^{i\,\underline{u}^t\tilde{\underline{\mu}}}$ as a power series,

$$E_{\underline{\mu}}E_{\underline{\underline{h}}|\underline{\mu}}[e^{i\,\underline{u}^t\tilde{\underline{\mu}}}] = E_{\underline{\mu}}\left[\sum_{k=0}^{\infty} \frac{(i)^k}{k!}\left(\sum_{\alpha=1}^{r} u_\alpha\tilde{\mu}_\alpha\right)^k\right]=\sum_{k=0}^{\infty} \frac{(i)^k}{k!}\, E_{\underline{\mu}}\left[\left(\sum_{\alpha=1}^{r} u_\alpha\tilde{\mu}_\alpha\right)^k\right] \tag{19}$$

The interchange of expectation operator and summation operator is justified by the fact that the power series converges for all real values of $\underline{u}$ and $\underline{\mu}$. Similarly, since, for fixed $\underline{\underline{h}}$, $\tilde{\underline{\mu}}$ has the Normal distribution with parameters $(\underline{0}, \underline{\underline{h}})$,

$$E_{\underline{\underline{h}}} E_{\underline{\mu}|\underline{\underline{h}}}[e^{i\,\underline{u}^t \tilde{\underline{\mu}}}] = E_{\underline{\underline{h}}}[\exp(-\tfrac{1}{2}\sum_{\alpha=1}^{r}\sum_{\beta=1}^{r} u_\alpha u_\beta \tilde{h}^{-1}_{\alpha\beta})]$$
$$= \sum_{k=0}^{\infty} \frac{(-\frac{1}{2})^k}{k!} E_{\underline{\underline{h}}}[(\sum_{\alpha=1}^{r}\sum_{\beta=1}^{r} u_\alpha u_\beta \tilde{h}^{-1}_{\alpha\beta})^k] \quad . \tag{20}$$

Since (19) and (20) are both power series expansions of the same function, coefficients of like powers of the variables u_α must be equal. We thus obtain the following identity, valid for all real $\underline{u}$:

$$\frac{1}{2^k k!} E_{\underline{\underline{h}}}[(\sum_{\alpha=1}^{r}\sum_{\beta=1}^{r} u_\alpha u_\beta \tilde{h}^{-1}_{\alpha\beta})^k] = \frac{1}{(2k)!} E_{\underline{\mu}}[(\sum_{\alpha=1}^{r} u_\alpha \tilde{\mu}_\alpha)^{2k}] \ . \tag{21}$$

In particular, for k=1,

$$\sum_{\alpha=1}^{r}\sum_{\beta=1}^{r} u_\alpha u_\beta E_{\underline{\underline{h}}}[\tilde{h}^{-1}_{\alpha\beta}] = \sum_{\alpha=1}^{r}\sum_{\beta=1}^{r} u_\alpha u_\beta E_{\underline{\mu}}[\tilde{\mu}_\alpha \tilde{\mu}_\beta] \tag{22}$$

and for k=2,

$$\sum_{\alpha,\beta,\gamma,\delta} u_\alpha u_\beta u_\gamma u_\delta E_{\underline{\underline{h}}}[\tilde{h}^{-1}_{\alpha\beta}\, \tilde{h}^{-1}_{\gamma\delta}] = \frac{1}{3} \sum_{\alpha,\beta,\gamma,\delta} u_\alpha u_\beta u_\gamma u_\delta E_{\underline{\mu}}[\tilde{\mu}_\alpha \tilde{\mu}_\beta \tilde{\mu}_\gamma \tilde{\mu}_\delta] \ . \tag{23}$$

The marginal distribution of $\tilde{\underline{\mu}}$ is multivariate Student with parameters $(\underline{0}, \nu\underline{\underline{V}}^{-1}, \nu)$. By noting that the Student distribution is the marginal distribution of $\tilde{\underline{\mu}}$ when $\tilde{\underline{\mu}}$ and the univariate random variable $\tilde{H}$ have the Normal-Gamma distribution,* it is seen that, provided $\nu > 4$,

$$E_{\underline{\mu}}[\tilde{\mu}_\alpha \tilde{\mu}_\beta \tilde{\mu}_\gamma \tilde{\mu}_\delta] = \int_{R_{\underline{\mu}}} \mu_\alpha \mu_\beta \mu_\gamma \mu_\delta \, f_S^{(r)}(\underline{\mu}|\underline{0}, \nu\underline{\underline{V}}^{-1}, \nu) d\underline{\mu}$$
$$= \int_0^\infty \int_{R_{\underline{\mu}}} \mu_\alpha \mu_\beta \mu_\gamma \mu_\delta \, f_N^{(r)}(\underline{\mu}|\underline{0}, \nu H\underline{\underline{V}}^{-1}) \, f_{\gamma 2}(H|1, \nu) d\underline{\mu}\, dH$$
$$= (V_{\alpha\beta}V_{\gamma\delta} + V_{\alpha\gamma}V_{\beta\delta} + V_{\alpha\delta}V_{\beta\gamma}) \int_0^\infty \frac{1}{\nu^2 H^2} f_{\gamma 2}(H|1, \nu) dH$$
$$= \frac{V_{\alpha\beta}V_{\gamma\delta} + V_{\alpha\gamma}V_{\beta\delta} + V_{\alpha\delta}V_{\beta\gamma}}{(\nu-2)(\nu-4)} \ . \tag{24}$$

---

*[3], p. 256.

Since (22) and (23) hold for all real $\underline{u}$,

$$E_{\underline{\underline{h}}}[\tilde{h}^{-1}_{\alpha\beta}] = E_{\underline{\mu}}[\tilde{\mu}_\alpha \tilde{\mu}_\beta] = \frac{V_{\alpha\beta}}{\nu-2} \quad , \qquad \nu > 2 \; , \tag{25}$$

which yields (13), and

$$E_{\underline{\underline{h}}}[\tilde{h}^{-1}_{\alpha\beta} \; \tilde{h}^{-1}_{\gamma\delta}] = \frac{1}{3} \, E_{\underline{\mu}}[\tilde{\mu}_\alpha \tilde{\mu}_\beta \tilde{\mu}_\gamma \tilde{\mu}_\delta] \tag{26}$$

which yields

$$\mathrm{cov}[\tilde{h}^{-1}_{\alpha\beta}, \; \tilde{h}^{-1}_{\gamma\delta}] = \frac{1}{3(\nu-2)(\nu-4)} \; [\frac{2(5-\nu)}{\nu-2} \; V_{\alpha\beta}V_{\gamma\delta}+V_{\alpha\gamma}V_{\beta\delta}+V_{\alpha\delta}V_{\beta\gamma}] \; . \tag{27}$$

If $\alpha=\gamma$, $\beta=\delta$, (27) reduces to

$$\mathrm{var}[\tilde{h}^{-1}_{\alpha\beta}] = \frac{1}{3(\nu-2)(\nu-4)} \; [\frac{8-\nu}{\nu-2} \; V^2_{\alpha\beta}+V_{\alpha\alpha}V_{\beta\beta}] \quad . \qquad \text{Q.E.D.} \tag{28}$$

<u>Corollary 1</u>. If $(\tilde{\underline{\mu}}, \tilde{\underline{\underline{h}}})$ has the Normal-Wishart distribution with parameters

$(\underline{m}, \underline{\underline{V}}, n, \nu)$ then

$$E\begin{bmatrix} \tilde{\underline{\mu}} \\ \\ \tilde{\underline{h}}^{-1} \end{bmatrix} = \begin{bmatrix} \underline{m} \\ \\ \frac{1}{(\nu-2)} \; \underline{V} \end{bmatrix} \tag{29}$$

where

$$\underline{V} = \{V_{11},\ldots,V_{1r},V_{22},\ldots,V_{rr}\} \tag{30}$$

and

$$V\begin{bmatrix} \tilde{\underline{\mu}} \\ \\ \tilde{\underline{h}}^{-1} \end{bmatrix} = \begin{bmatrix} \frac{1}{n(\nu-2)} \; \underline{\underline{V}} & \underline{\underline{0}} \\ \\ \underline{\underline{0}} & \underline{\underline{W}}(\underline{\underline{V}}) \end{bmatrix} \tag{31}$$

<u>Proof</u>: Noting that $\tilde{\underline{\underline{h}}}^{-1}$ is a function only of $\tilde{\underline{\underline{h}}}$ and that the conditional expectation of $\tilde{\underline{\mu}}$ given $\underline{\underline{h}}$ is functionally independent of $\underline{\underline{h}}$, the corollary is an immediate consequence of Theorems 1 and 2. Q.E.D.

<u>Theorem 3</u>. If the mr x 1 vector $\tilde{\underline{\pi}}$ has the non-degenerate generalized multivariate Student distribution

$$f^{(mr)}_{SG}(\tilde{\underline{\pi}}|\underline{p}, \underline{\underline{V}}, \underline{\underline{\epsilon}}, \nu)$$

then

$$E[\tilde{\underline{\pi}}] = \underline{p} \tag{32}$$

and the variance-covariance matrix of $\tilde{\underline{\pi}}$ is

$$V[\tilde{\underline{\pi}}] = \frac{1}{(\nu-2)} \underline{\underline{\epsilon}} \otimes \underline{\underline{V}}^{-1} \quad , \qquad \nu > 2 \ , \tag{33}$$

where $\otimes$ denotes the Kronecker product.

<u>Proof:</u> The non-degenerate generalized multivariate Student distribution is obtained in [2] as the marginal density function of $\tilde{\underline{\pi}}$ when $\tilde{\underline{\pi}}$ and the m x m matrix $\tilde{\underline{\underline{h}}}$ have the joint density function.

$$D(\underline{\pi}, \underline{\underline{h}}|\underline{p}, \underline{\underline{V}}, \underline{\underline{\epsilon}}, \nu) = f_N^{(mr)}(\underline{\pi}|\underline{p}, \underline{\underline{h}} \otimes \underline{\underline{V}})\ f_W^{(m)}(\underline{\underline{h}}|\underline{\underline{\epsilon}}, \nu) \quad .$$

Thus

$$E_{\underline{\pi}}[\tilde{\underline{\pi}}] = E_{\underline{\pi}} E_{\underline{\underline{h}}|\underline{\pi}}[\tilde{\underline{\pi}}] = E_{\underline{\underline{h}}} E_{\underline{\pi}|\underline{\underline{h}}}[\tilde{\underline{\pi}}] = \underline{p} \quad . \tag{34}$$

$$V[\tilde{\underline{\pi}}] = E_{\underline{\underline{h}}} E_{\underline{\pi}|\underline{\underline{h}}}[(\tilde{\underline{\pi}}-\underline{p})(\tilde{\underline{\pi}}-\underline{p})^t] = E_{\underline{\underline{h}}}[\tilde{\underline{\underline{h}}}^{-1} \otimes \underline{\underline{V}}^{-1}] \ . \tag{35}$$

Since the marginal distribution of $\tilde{\underline{\underline{h}}}$ is Wishart with parameters $(\underline{\underline{\epsilon}}, \nu)$, Theorem 2 yields

$$V[\tilde{\underline{\pi}}] = \frac{1}{\nu-2} \underline{\underline{\epsilon}} \otimes \underline{\underline{V}}^{-1} \quad , \tag{36}$$

provided $\nu > 2$. Q.E.D.

<u>Corollary 2</u>. If $(\tilde{\underline{\pi}}, \tilde{\underline{\underline{h}}})$ has the Normal-Wishart distribution,

$$f_N^{(mr)}(\underline{\pi}|\underline{p}, \underline{\underline{h}} \otimes \underline{\underline{V}})\ f_W^{(m)}(\underline{\underline{h}}|\underline{\underline{\epsilon}}, \nu)$$

then

$$E\begin{bmatrix} \tilde{\underline{\pi}} \\ \tilde{\underline{h}}^{-1} \end{bmatrix} = \begin{bmatrix} \underline{p} \\ \frac{1}{(\nu-2)} \underline{\epsilon} \end{bmatrix} \quad , \qquad \nu > 2 \ , \tag{37}$$

where

$$\underline{\epsilon} = \{\epsilon_{11}, \ldots, \epsilon_{1m}, \epsilon_{22}, \ldots, \epsilon_{mm}\} \quad , \tag{38}$$

and

$$V\begin{bmatrix} \tilde{\underline{\pi}} \\ \tilde{\underline{h}}^{-1} \end{bmatrix} = \begin{bmatrix} \frac{1}{(\nu-2)} \underline{\underline{\epsilon}} \otimes \underline{\underline{V}}^{-1} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{W}}(\underline{\underline{\epsilon}}) \end{bmatrix} , \qquad \nu > 4 \quad . \tag{39}$$

3. Radar-Tracking Model

In order to illustrate preposterior and prior-posterior analysis for the independent multivariate Normal process when uncertainty exists about the mean and the precision, we consider in this section a simple radar-tracking model. The Bayesian analysis follows [1].

Let a single aircraft, on a fixed course and speed at a fixed altitude, be tracked by a single radar. Let

$$\underline{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

be the unknown velocity of the aircraft, with east-west component v_1 and north-south component v_2.

If

$$\underline{\xi}^{(k)} = \begin{bmatrix} \xi_1^{(k)} \\ \xi_2^{(k)} \end{bmatrix} \quad , \quad k=1,2,\ldots$$

is the position of the aircraft at the time of the kth radar scan and if

$$\underline{\tilde{\mu}}^{(k)} = \begin{bmatrix} \tilde{\mu}_1^{(k)} \\ \tilde{\mu}_2^{(k)} \end{bmatrix} \quad , \quad k=1,2,\ldots$$

is the radar error in measuring aircraft position on the kth scan (a random variable), then the radar-derived velocity on the kth scan is, letting $\underline{\tilde{e}}^{(k)} = \underline{\tilde{\mu}}^{(k)} - \underline{\tilde{\mu}}^{(k-1)}$,

$$\underset{\sim}{\tilde{x}}^{(k)} = (\underset{\sim}{\xi}^{(k)} + \underset{\sim}{\tilde{\mu}}^{(k)}) - (\underset{\sim}{\xi}^{(k-1)} + \underset{\sim}{\mu}^{(k-1)}) = \underset{\sim}{v} + \underset{\sim}{\tilde{e}}^{(k)} \tag{40}$$

since

$$\underset{\sim}{v} = \underset{\sim}{\xi}^{(k)} - \underset{\sim}{\xi}^{(k-1)} .$$

If the radar errors, $\underset{\sim}{\tilde{\mu}}^{(k)}$, $k=1,2,\ldots$, are assumed to be independent, identically distributed random variables, each having the bivariate Normal distribution with mean $\underset{\sim}{0}$, then the radar-derived velocities, $\underset{\sim}{\tilde{x}}^{(k)}$, are independent, identically distributed random variables, each with the bivariate Normal distribution with mean $\underset{\sim}{v}$ and precision $\underset{\approx}{h}$, neither of which is known with certainty. While it is true that the variance-covariance matrix for radar error can be derived from the theory of radar signals, uncertainty about $\underset{\approx}{h}$ is introduced by numerous factors in the tracking system, including weather conditions, physical condition of the radar equipment, and random delays in processing radar data.

Let us initially assume that prior information is negligible. Inspection of equation (7) of [1] shows that, if the parameters of the posterior distribution are to be determined only by sample information, then the following values must be assigned to the prior parameters:

$$n'=0 \quad , \quad \nu'=0 \quad , \quad \underset{\sim}{m}'=\underset{\sim}{0} \quad , \quad \underset{\approx}{V}'=\underset{\approx}{0} \quad . \tag{41}$$

Then, if $n > 2$ independent radar observations of target velocity are made, yielding the statistics n, ν, $\underset{\sim}{m}$, and $\underset{\approx}{V}$, the parameters of the posterior Normal-Wishart distribution are, since $\delta'=0$, $\delta=1$, and $\phi=0$,

$$n''=n \quad , \quad \underset{\sim}{m}''=\underset{\sim}{m} \quad , \quad \nu''=n-1 \quad , \quad \underset{\approx}{V}''=\underset{\approx}{V} \quad . \tag{42}$$

Now suppose the following five radar observations of target velocity are obtained,

$$\underset{\sim}{x}^{(1)} = \begin{bmatrix} 539 \\ 309 \end{bmatrix} , \quad \underset{\sim}{x}^{(2)} = \begin{bmatrix} 514 \\ 290 \end{bmatrix} , \quad \underset{\sim}{x}^{(3)} = \begin{bmatrix} 528 \\ 300 \end{bmatrix}$$

$$\underset{\sim}{x}^{(4)} = \begin{bmatrix} 521 \\ 294 \end{bmatrix} , \quad \underset{\sim}{x}^{(5)} = \begin{bmatrix} 524 \\ 310 \end{bmatrix} . \tag{43}$$

The data upon which this sample is based is presented in Appendix B. Upon calculating the sample statistics, ν, n, $\underline{m}$ and $\underline{\underline{V}}$, and substituting them into (42), the posterior distribution is found to be nondegenerate Normal-Wishart, with parameters

$$n''=5 \quad , \quad \nu''=4$$

$$\underline{m}''= \underline{m} = \begin{bmatrix} 525.2 \\ 300.6 \end{bmatrix} \quad , \quad \underline{\underline{V}}''= \underline{\underline{V}} = \begin{bmatrix} 342.6 & 249.3 \\ 249.3 & 315.4 \end{bmatrix} . \tag{44}$$

The actual mean vector, $\underline{w}$, and variance matrix, $\underline{\underline{S}}$, used in constructing the sample (43) were

$$\underline{w} = \begin{bmatrix} 520 \\ 300 \end{bmatrix} \quad , \quad \underline{\underline{S}} = \begin{bmatrix} 120 & 70 \\ 70 & 90 \end{bmatrix} ,$$

equivalent to a target on course 060° at a speed of 600 knots, with a standard deviation of 10.95 knots in the east-west component of radar-derived velocity and a standard deviation of 9.48 knots in the north-south component. The correlation coefficient between $\tilde{x}_1$ and $\tilde{x}_2$ is 0.673.

With negligible prior information and the sample (43), the posterior expectation of $\underline{\tilde{v}}$ and $\underline{\underline{\tilde{h}}}^{-1}$ is, using Corollary 1,

$$E[\underline{\tilde{v}}] = \underline{m}'' = \begin{bmatrix} 525.2 \\ 300.6 \end{bmatrix} \tag{45}$$

$$E[\underline{\underline{\tilde{h}}}^{-1}] = \frac{1}{\nu''-2} \underline{\underline{V}}'' = \tfrac{1}{2}\underline{\underline{V}}'' = \begin{bmatrix} 171.3 & 124.6 \\ 124.6 & 157.7 \end{bmatrix} . \tag{46}$$

Let us now construct a prior distribution which is "tight" on $\underline{\underline{\tilde{h}}}$, but "loose" on $\underline{\tilde{v}}$. This procedure might be desirable, for example, when the variance-covariance matrix of the rectangular components of error, $\tilde{e}_1$ and $\tilde{e}_2$, is virtually independent of target velocity.

In assigning a prior distribution to $(\underline{\tilde{v}}, \underline{\underline{\tilde{h}}})$, there are

$$N = r + \frac{r(r+1)}{2} + 1 + 1 = \tfrac{1}{2}(r^2+3r+4)$$

degrees of freedom in the prior parameters $\underline{m}'$, $\underline{\underline{V}}'$, n', ν'. Using the results of Theorems 1 and 2 and Corollary 1, the following systematic procedure for assigning prior

parameters is suggested.

a. The decision-maker states the mean, $\bar{\underline{\underline{h}}}^{-1}$, of his marginal prior on $\tilde{\underline{\underline{h}}}^{-1}$. Using Theorem 2, $\underline{\underline{V}}'$ is obtained in terms of $\bar{\underline{\underline{h}}}^{-1}$ and the, as yet, unspecified parameter ν':

$$\underline{\underline{V}}' = (\nu'-2)\bar{\underline{\underline{h}}}^{-1} . \tag{47}$$

If the decision-maker desires, he may, alternatively, state the mean, $\bar{\underline{\underline{h}}}$, of his marginal prior on $\tilde{\underline{\underline{h}}}$; then, using (2),

$$\underline{\underline{V}}'^{-1} = \frac{1}{(\nu'+r-1)} \bar{\underline{\underline{h}}} . \tag{48}$$

Most individuals, however, find it easier to make probability statements about the variance-covariance matrix, $\tilde{\underline{\underline{h}}}^{-1}$, rather than its inverse, and we shall use (47).

b. The decision-maker states the mean, $\bar{\underline{v}}$, of his marginal prior on $\tilde{\underline{v}}$. The prior parameter $\underline{m}'$ is

$$\underline{m}' = \bar{\underline{v}} \tag{49}$$

c. Having assigned values to $\underline{m}'$ and $\underline{\underline{V}}'$ in such a way that the mean of the Normal-Wishart prior on $(\tilde{\underline{v}}, \tilde{\underline{\underline{h}}})$ is fixed, there remain two degrees of freedom by which to further characterize the prior. Ideally, one would desire to have at least $r + \frac{1}{2}r(r+1)$ degrees of freedom remaining, allowing the decision-maker to incorporate into the prior on $(\tilde{\underline{v}}, \tilde{\underline{\underline{h}}})$ the variances of his marginal priors on $\tilde{\underline{v}}$ and $\tilde{\underline{\underline{h}}}^{-1}$. However, the Normal-Wishart distribution is extremely parsimonious in this respect.

Two possible alternatives suggest themselves. The first is to obtain from the decision-maker his marginal variance for one element, $\tilde{v}_i$, of $\tilde{\underline{v}}$ and his marginal variance for one element, $\tilde{h}^{-1}_{ij}$, of $\tilde{\underline{\underline{h}}}^{-1}$ (or, equivalently, an element of $\tilde{\underline{\underline{h}}}$). These variances are used to fix n' and ν', implying variances and covariances for the remaining elements of $\tilde{\underline{v}}$ and $\tilde{\underline{\underline{h}}}$. If these implied variances and covariances are in tolerable agreement with the decision-maker's prior knowledge, then n' and ν' are used as parameters for the prior

distribution of $(\tilde{\underline{v}}, \tilde{\underline{\underline{h}}})$. Precisely what is meant by "tolerable" must be defined in terms of the terminal cost structure of the problem.

Another method of procedure is to choose n' and ν' so as to minimize either the maximum absolute deviation or the sum of squared deviations between the variances and covariances of $\tilde{\underline{v}}$ and $\tilde{\underline{\underline{h}}}$ implied by n' and ν' and those of the decision-maker. This procedure may or may not be justified, depending upon the terminal cost structure of the problem. The first method is developed here, resulting in equations (51) and (54). These equations can also be used to develop the least-squares or minimum absolute deviation selection of n' and ν'.

If the variance of $\tilde{h}^{-1}_{ii}$ in the decision-maker's marginal prior on $\tilde{\underline{\underline{h}}}^{-1}$ is σ_{ii} then, by Theorem 2,

$$\sigma_{ii} = \frac{2V'^{2}_{ii}}{(\nu'-2)^2(\nu'-4)} = \frac{2}{\nu'-4}\,(\bar{h}^{-1}_{ii})^2 \tag{50}$$

which yields

$$\nu' = 4 + \frac{2}{\sigma_{ii}}\,(\bar{h}^{-1}_{ii})^2 \ . \tag{51}$$

If (48) were used to fix $\underline{\underline{V}}'$ in terms of ν', then it follows from (6) that

$$\nu' = \frac{1}{\sigma_{ij}}\,(\bar{h}^{2}_{ij} + \bar{h}_{ii}\,\bar{h}_{jj}) - r + 1 \ , \tag{52}$$

where σ_{ij} is the variance of $\tilde{h}_{ij}$ in the decision-maker's marginal prior on $\tilde{\underline{\underline{h}}}$.

Since $\tilde{\underline{v}}$ has the multivariate Student distribution with parameters $(\underline{m}', n'\nu'\underline{\underline{V}}'^{-1}, \nu')$, it follows that the marginal variance-covariance matrix of $\tilde{\underline{v}}$ is

$$V[\tilde{\underline{v}}] = \frac{1}{n'(\nu'-2)}\,\underline{\underline{V}}' = \frac{1}{n'}\,\bar{\underline{\underline{h}}}^{-1} \ . \tag{53}$$

If s_i is the variance of $\tilde{v}_i$ in the decision-maker's marginal prior on $\underline{v}$, then

$$n' = \frac{\bar{h}^{-1}_{ii}}{s_i} \ . \tag{54}$$

An alternative method of fixing n' and ν' uses the squared coefficients of variation of the decision-makers marginal priors. Let C^h_{ii} be the squared coefficient

of variation of $\tilde{h}^{-1}_{ii}$ and let C^v_j be the squared coefficient of variation of $\tilde{v}_j$. Then

$$C^h_{ii} = \frac{2V'^2_{ii}}{(\nu'-2)^2(\nu'-4)} \cdot \frac{(\nu'-2)^2}{V'^2_{ii}} = \frac{2}{\nu'-4} \ , \tag{55}$$

$$C^v_j = \frac{V'_{jj}}{m'^2_j \; n'(\nu'-2)} = \frac{\bar{h}^{-1}_{jj}}{m'^2_j \; n'} \ , \tag{56}$$

yielding

$$\nu' = \frac{2}{C^h_{ii}} + 4 \ , \tag{57}$$

$$n' = \frac{\bar{h}^{-1}_{jj}}{m'^2_j \; C^v_j} \ . \tag{58}$$

Of course, using the squared coefficient of variation does not eliminate the inherent difficulty of fitting two parameters to $r+\frac{1}{2}r(r+1)$ prior variance, but use of (57) and (58) may simplify the procedure of fitting parameters.

To illustrate the use of the above procedure for assigning values to the prior parameters, consider the following example. Let us a priori assume the target will have that velocity which is of the greatest threat--i.e., assume the target is on a course for the task force center and is travelling at maximum speed. Suppose, for the present target position, this velocity is 650 knots on course 045° (cf. Figure 1).

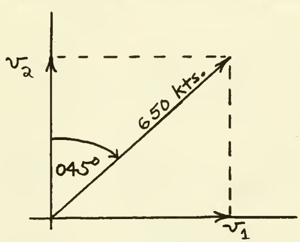

Figure 1

Then

$$\bar{v}_1 = \bar{v}_2 = 650 \cos 45^\circ = 459.6 \text{ knots.}$$

Let the expected variance-covariance matrix be

$$\underset{=}{h}^{-1} = \begin{bmatrix} 110 & 55 \\ 55 & 110 \end{bmatrix} ,$$

which implies a standard deviation of 10.49 knots in both $\tilde{v}_1$ and $\tilde{v}_2$, with a correlation coefficient of 0.5. To be tight on $\tilde{\underset{=}{h}}$, let the prior squared coefficient of variation of both h_{11}^{-1} and h_{22}^{-1} be 0.05. To be loose on $\tilde{\underset{-}{v}}$, let the prior standard deviation in both $\tilde{v}_1$ and $\tilde{v}_2$ be 150 knots. Then, using (47), (49), (54), and (57), the prior parameters are

$$\underset{-}{m}' = \begin{bmatrix} 459.6 \\ 459.6 \end{bmatrix} , \qquad \nu' = \frac{2}{0.05} + 4 = 44$$

$$\underset{=}{V}' = (\nu'-2) \begin{bmatrix} 110 & 55 \\ 55 & 110 \end{bmatrix} = \begin{bmatrix} 4620 & 2310 \\ 2310 & 4620 \end{bmatrix} \tag{59}$$

$$n' = \frac{110}{22500} = 0.00489 .$$

If the sample (43) is observed, with the sample statistics

$$n=5 \quad , \quad \nu=3$$

$$\underset{-}{m} = \begin{bmatrix} 525.2 \\ 300.6 \end{bmatrix} , \qquad \underset{=}{V} = \begin{bmatrix} 342.6 & 249.3 \\ 249.3 & 315.4 \end{bmatrix} , \tag{60}$$

use of equation (7) of [1] yields the following posterior parameters:

$$n'' = 5.00489 \quad , \quad \nu'' = 49 \tag{61}$$

$$\underset{-}{m}'' = \frac{1}{5.00489} \left(\begin{bmatrix} 2.25 \\ 2.25 \end{bmatrix} + \begin{bmatrix} 2626.00 \\ 1503.00 \end{bmatrix} \right) = \begin{bmatrix} 525.1 \\ 300.8 \end{bmatrix} \tag{62}$$

$$\underset{=}{V}'' = \begin{bmatrix} 4620 & 2310 \\ 2310 & 4620 \end{bmatrix} + \begin{bmatrix} 342.6 & 249.3 \\ 249.3 & 315.4 \end{bmatrix}$$

$$+ \begin{bmatrix} 1032.9 & 1032.9 \\ 1032.9 & 1032.9 \end{bmatrix} + \begin{bmatrix} 1,379,175.2 & 789,375.6 \\ 789,375.6 & 451,801.8 \end{bmatrix}$$

$$- \begin{bmatrix} 1,379,998.4 & 790,522.8 \\ 790,522.8 & 452,845.7 \end{bmatrix} \tag{63}$$

$$= \begin{bmatrix} 5172.3 & 2445.0 \\ 2445.0 & 4924.4 \end{bmatrix}$$

The posterior expected values of $\underset{\sim}{\tilde{v}}$ and $\underline{\underline{\tilde{h}}}^{-1}$ are

$$E[\underset{\sim}{\tilde{v}}] = \underline{m}'' = \begin{bmatrix} 525.1 \\ 300.8 \end{bmatrix}$$

$$E[\underline{\underline{\tilde{h}}}^{-1}] = \frac{1}{47} \underline{\underline{V}}'' = \begin{bmatrix} 110.0 & 52.0 \\ 52.0 & 104.8 \end{bmatrix} .$$

The actual mean vector and covariance matrix of the population from which the sample (43) was drawn is

$$\underline{w} = \begin{bmatrix} 520 \\ 300 \end{bmatrix} , \qquad \underline{\underline{S}} = \begin{bmatrix} 120 & 70 \\ 70 & 90 \end{bmatrix} .$$

Note that being diffuse on $\underline{\tilde{v}}$, through an extremely small n', effectively negated the prior parameter $\underline{m}'$, so that $\underline{m}''$ essentially reflects the sample mean. On the other hand, being tight on $\underline{\underline{\tilde{h}}}$ through a large ν', the sample had little effect on $E[\underline{\underline{\tilde{h}}}^{-1}$

We next illustrate the preposterior analysis of the multivariate Normal process. There is a cost associated with the collection of radar observations, since, with each additional observation, the target moves closer to the force center and therefore poses an increased threat. Assume, for simplicity, that the cost, $c_s(n)$, of making n radar observations on the target before firing, is linear:

$$c_s(n) = an \quad , \qquad n=0,1,2,\ldots \quad . \tag{64}$$

On the other hand, the more observations collected before firing, the greater the probability of destroying the target when it is fired upon. Assume that, if $\hat{\underline{v}}$ is the estimated velocity set into the fire control mechanism when the target is fired upon, the cost of firing after n observations is quadratic:

$$c_e(n) = K[(v_1-\hat{v}_1)^2 + (v_2-\hat{v}_2)^2] \quad . \tag{65}$$

Here $\underline{v}$ is the target's actual velocity and $c_e(n)$ is the expected target threat after the weapon is fired. K is a constant. Regarding $\underline{v}$ as a random variable, it is shown on p. 188 of [3] that, if the estimate $\hat{\underline{v}}$ is chosen so as to minimize $E_{\underline{v}}[\tilde{c}_e(n)]$, then

$$\hat{\underline{v}} = E[\tilde{\underline{v}}] \tag{66}$$

and the expected cost of firing is then

$$E_{\underline{v}}[\tilde{c}_e(n)] = K(\mathrm{var}[\tilde{v}_1] + \mathrm{var}[\tilde{v}_2] \; . \tag{67}$$

After a sample of size n is drawn, the posterior marginal distribution of $\tilde{\underline{v}}$ is multivariate Student with parameters $(\underline{m}'', \nu''n''\underline{\underline{V}}''^{-1}, \nu'')$ and covariance matrix

$$V[\tilde{\underline{v}}] = \frac{\nu''}{\nu''-2}\,\frac{1}{\nu''n''}\,\underline{\underline{V}}'' = \frac{1}{n''(\nu''-2)}\,\underline{\underline{V}}'' \quad . \tag{68}$$

Applying equation (5-28) of [3], if $E_{m,V}[\cdot]$ denotes the expectation operator relative to the distribution of the sample statistic $(\tilde{\underline{m}}, \tilde{\underline{\underline{V}}})$, then, before the sample is drawn,

$$E_{m,V}[\tilde{\check{\underline{\underline{v}}}}''] = \check{\underline{\underline{v}}}' - V_{m,V}[\tilde{\underline{m}}''] \quad . \tag{69}$$

Here $\tilde{\check{\underline{\underline{v}}}}''$ is the posterior variance-covariance matrix of $\tilde{\underline{v}}$ and is a random variable before the sample is drawn; $\check{\underline{\underline{v}}}'$ is the prior variance-covariance matrix of $\tilde{\underline{v}}$ and $V_{m,V}[\cdot]$ denotes the variance-covariance matrix of a random vector relative to the distribution of $(\tilde{\underline{m}}, \tilde{\underline{\underline{V}}})$. Since, before the sample is drawn, $\tilde{\underline{m}}''$ has the multivariate Student distribution* with parameters $(\underline{m}', (n''/n)^2\underline{\underline{H}}_{\nu'}, \nu')$, where

$$\underline{\underline{H}}_{\nu'} = \nu'\,\frac{n'n}{n''}\,\underline{\underline{V}}'^{-1} \quad ,$$

(69) can be written as

$$E_{m,V}[\tilde{\check{\underline{\underline{v}}}}''] = \frac{\nu'}{\nu'-2}\;\frac{1}{\nu'n'}\;\underline{\underline{V}}' - \frac{\nu'}{\nu'-2}\;\left(\frac{n}{n''}\right)^2\;\frac{n''}{\nu'nn'}\;\underline{\underline{V}}' = \frac{1}{n''(\nu'-2)}\;\underline{\underline{V}}' \tag{70}$$

and (67) becomes, upon taking the expected value of $E_{\underline{v}}[\tilde{c}_e(n)]$ over the distribution of $(\tilde{\underline{m}}, \tilde{\underline{\underline{V}}})$,

---

*[1], pp. 18 and 19.

$$E[\tilde{c}_e(n)] = \frac{K}{n''(\nu'-2)} (V'_{11} + V'_{22}) \; . \tag{71}$$

If n observations are taken before firing, let the total expected cost be $\overline{C}(n)$. Then

$$\overline{C}(n) = an + \frac{K}{(n'+n)(\nu'-2)} (V'_{11} + V'_{22}) \; . \tag{72}$$

Differentiating (72) with respect to n and setting the result equal to zero gives the following equation for the optimal number of observations, n^*:

$$n^* = \sqrt{\frac{K(V'_{11}+V'_{22})}{a(\nu'-2)}} - n' \; . \tag{73}$$

If $n^* < 0$, no observations are to be taken. If n^* is not an integer, choose the integer nearest n^* whic

For a Normal-Wishart prior distribution on $(\tilde{\underline{v}}, \tilde{\underline{\underline{h}}})$ with parameters given by (59), the optimal sample size is

$$n^* = \sqrt{\frac{2(4620)K}{42a}} - 0.00489$$

$$= 14.83 \sqrt{\frac{K}{a}} - 0.00489 \; .$$

If K=8, a=2, then

$$n^* = 29.66 \doteq 30 \; . \tag{74}$$

We now consider a prior distribution on $(\tilde{\underline{v}}, \tilde{\underline{\underline{h}}})$ which is tight on $\tilde{\underline{v}}$ and loose on $\tilde{\underline{\underline{h}}}$. This situation is unlikely in the radar-tracking model, since it amounts to having a small prior variance on each of the elements of $\tilde{\underline{v}}$, the target velocity, while having a large variance on each of the elements of $\underline{\underline{h}}$, which reflects the radar error. Nevertheless, some instructive results follow, so we shall illustrate this case.

Again using the expected target velocity and error variance,

$$\overline{\underline{v}} = \begin{bmatrix} 459.6 \\ 459.6 \end{bmatrix} , \qquad \overline{\underline{\underline{h}}}^{-1} = \begin{bmatrix} 110 & 55 \\ 55 & 110 \end{bmatrix} ,$$

we now assume the prior squared coefficient of variation of both $\tilde{h}_{11}^{-1}$ and $\tilde{h}_{22}^{-1}$ to be 0.5, while the prior standard deviation of both $\tilde{v}_1$ and $\tilde{v}_2$ is 3 knots. Applying (47), (49), (54), and (57), we obtain the prior parameter

$$\underline{m}' = \begin{bmatrix} 459.6 \\ 459.6 \end{bmatrix} , \qquad \nu' = \frac{2}{0.5} + 4 = 8$$

$$\underline{\underline{V}}' = 6 \begin{bmatrix} 110 & 55 \\ 55 & 110 \end{bmatrix} = \begin{bmatrix} 660 & 330 \\ 330 & 660 \end{bmatrix} \tag{75}$$

$$n' = \frac{110}{9} = 12.2$$

Using (73), the optimal sample size is

$$n* = 14.83 \sqrt{\frac{K}{a}} - 12.2$$

which, in the case K=8, a=2, is

$$n* = 17.46 \doteq 18 \quad . \tag{76}$$

Comparison with (74) shows that, being tighter on $\tilde{\underline{v}}$ (as reflected in n'), fewer observations are required to attain a reasonable probability of destroying the target.

If the sample (43) of size n=5 is drawn, with the statistics (60), application of equation (7) of [1] yields the following posterior parameters:

$$n''=17.2 \qquad \nu''=13 \tag{77}$$

$$\underline{m}'' = \frac{1}{17.2} \left(\begin{bmatrix} 5607.1 \\ 5607.1 \end{bmatrix} + \begin{bmatrix} 2626.0 \\ 1503.0 \end{bmatrix} \right) = \begin{bmatrix} 478.7 \\ 413.4 \end{bmatrix} \tag{78}$$

$$\underline{\underline{V}}'' = \begin{bmatrix} 660 & 330 \\ 330 & 660 \end{bmatrix} + \begin{bmatrix} 342.6 & 249.3 \\ 249.3 & 315.4 \end{bmatrix}$$

$$+ \begin{bmatrix} 2{,}577{,}032.4 & 2{,}577{,}032.4 \\ 2{,}577{,}032.4 & 2{,}577{,}032.4 \end{bmatrix}$$

$$+ \begin{bmatrix} 1{,}379{,}175.2 & 789{,}375.6 \\ 789{,}375.6 & 451{,}801.8 \end{bmatrix}$$

$$- \begin{bmatrix} 3,941,443.5 & 3,403,786.8 \\ 3,403,786.8 & 2,939,472.4 \end{bmatrix}$$

$$= \begin{bmatrix} 15,766.7 & -36,799.5 \\ -36,799.5 & 90,337.2 \end{bmatrix}, \qquad (79)$$

The posterior expected values of $\underset{-}{\tilde{v}}$ and $\underset{=}{\tilde{h}}^{-1}$ are

$$E[\underset{-}{\tilde{v}}] = \underset{-}{m}'' = \begin{bmatrix} 478.7 \\ 413.4 \end{bmatrix}$$

and

$$E[\underset{=}{\tilde{h}}''^{-1}] = \frac{1}{11} \underset{=}{V}'' = \begin{bmatrix} 1433.3 & -3345.4 \\ -3345.4 & 8212.5 \end{bmatrix} .$$

The actual mean and covariance of the process from which the sample* (43) was drawn is

$$\underset{-}{w} = \begin{bmatrix} 520 \\ 300 \end{bmatrix} , \qquad \underset{=}{S} = \begin{bmatrix} 120 & 70 \\ 70 & 90 \end{bmatrix} .$$

It is clear that being tight on $\underset{-}{\tilde{v}}$ has prevented the sample from being as influential on the posterior parameter $\underset{-}{m}''$ as in the case of the prior with $n'=0.00489$. Of greater significance is the fact that being jointly tight on $\underset{-}{\tilde{v}}$ and loose on $\underset{=}{\tilde{h}}$ has produced a striking example of the effect discussed in [1], p. 8. By choosing $n'=12.2$ and $\underset{-}{m}' = \{459.6, 459.6\}$, certain characteristics were imputed to the marginal prior on $\underset{=}{\tilde{h}}$ which were not reflected in our choice of $\underset{=}{V}'$ and ν'. These characteristics did not become evident until the sample statistics were combined with the prior distribution. In this connection it is of interest to note that the prior variance of both $\tilde{h}_{11}^{-1}$ and $\tilde{h}_{22}^{-1}$ is 6.05×10^3, while the posterior variances are

$$\mathrm{var}[\tilde{h}_{11}^{-1}] = 4.57 \times 10^5 , \qquad \mathrm{var}[\tilde{h}_{22}^{-1}] = 1.50 \times 10^7 .$$

---

*The sample mean and unbiassed sample variance of (43) are

$$\underset{-}{\bar{w}} = \begin{bmatrix} 525.2 \\ 300.6 \end{bmatrix} , \qquad \underset{=}{\bar{S}} = \tfrac{1}{4} \underset{=}{V} = \begin{bmatrix} 85.6 & 62.3 \\ 62.3 & 78.8 \end{bmatrix}$$

Some comments about the radar-tracking model itself are perhaps in order. The model was termed a simple one because it concerned itself only with target velocity and did not permit the target to maneuver. These restrictions can probably be removed by an extension of the analysis developed here. A more serious defect of the model is the assumption that the radar error, $\underline{\tilde{\mu}}^{(k)}$, when expressed in rectangular coordinates, is characterized by a stationary distribution. The theory of radar signals indicates that it is reasonable to assume the distribution of the radar error, when expressed in polar coordinates, $\tilde{\mu}_\rho^{(k)}$ and $\tilde{\mu}_\theta^{(k)}$, is stationary. But, expressing the rectangular components of error in terms of the polar components,

$$
\begin{aligned}
\tilde{\mu}_1^{(k)} &= \sqrt{(\tilde{\mu}_\rho^{(k)})^2 + (\tilde{\mu}_\theta^{(k)})^2} \qquad \cos(\Phi^{(k)} + \theta^{(k)}) \\
\tilde{\mu}_2^{(k)} &= \sqrt{(\tilde{\mu}_\rho^{(k)})^2 + (\tilde{\mu}_\theta^{(k)})^2} \qquad \sin(\Phi^{(k)} + \theta^{(k)}) \quad ,
\end{aligned}
\tag{80}
$$

where $\theta^{(k)}$ is the target's angular position on the kth scan and

$$\Phi^{(k)} = \tan^{-1} \frac{\mu_\theta^{(k)}}{\mu_\rho^{(k)}} \quad ,$$

it is seen that the joint distribution of $\{\tilde{\mu}_1^{(k)}, \tilde{\mu}_2^{(k)}\}$ will, in general, vary with the target's position.* In a computer-controlled tracking system, working space in memory and computing time are conserved by using a rectangular coordinate system, so it is desirable to remove this restriction. Unfortunately, the present multivariate Bayesian theory does not extend to the case of non-stationary data generating processes.

---

*Note that, in the case of targets flying directly toward force center, which are the most threatening targets, $\theta(k)$ is virtually independent of k and the assumption of a stationary distribution is valid.

4. Leontief's Static Inter-Industry Model

The Leontief static inter-industry model is a system of simultaneous equations in which the technological coefficients may be regarded as unknown, but fixed, parameters which must be determined from observations of consumer demand and industrial output. We shall use the prior-posterior analysis developed by Ando and Kaufman [2] for the reduced form data generating process. To simplify numerical computation, a two-industry economy is considered.

Let

x_i = annual output of industry i (i=1,2)

b_i = amount per year of the product of industry i demanded by all sectors of the economy other than industries 1 and 2 (e.g., government demand, consumer demand) (i=1,2)

a_{ij} = number of units of product i required to produce one unit of product j (technological coefficients). (i,j=1,2)

Then the annual demand for product i is

$$x_i = a_{ii}x_i + a_{ij}x_j + b_i \quad , \qquad i,j=1,2$$

which yields the system of simultaneous equations linear in the x_i,

$$\begin{aligned} (1-a_{11})\, x_1 - a_{12}x_2 &= b_1 \\ -a_{21}\; x_1 + (1-a_{22})x_2 &= b_2 \; . \end{aligned} \tag{81}$$

Let $\underline{\underline{A}} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\underline{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

and

$$\underline{\underline{\Pi}} = \underline{\underline{I}} - \underline{\underline{A}} \quad .$$

Then the equations (81) may be written

$$\underline{\underline{\Pi}}\;\underline{x} = \underline{b} \quad . \tag{82}$$

Equation (82) can be used to determine the annual output of industries 1 and 2 if $\underline{b}$ is a forecast of consumer demand and the technological matrix $\underline{\underline{A}}$ is known. Alternatively, (82) can be used to estimate $\underline{\underline{\Pi}}$ (and, therefore, $\underline{\underline{A}}$) from observations $\underline{x}^{(k)}$ and $\underline{b}^{(k)}$, of annual industrial output and consumer demand. Since $\underline{x}^{(k)}$, the output

in the kth year, is based on a forecast of $\underline{b}^{(k)}$, (82) will not be satisfied exactly by $\underline{x}^{(k)}$ and $\underline{b}^{(k)}$. Let the anticipated demand for the kth year be $\underline{\beta}^{(k)}$,

$$\underline{\beta}^{(k)} = \underline{\tilde{b}}^{(k)} - \underline{\tilde{v}}^{(k)} , \tag{83}$$

where $\underline{\tilde{b}}^{(k)}$ is the actual demand in the kth year (a random variable prior to its observation) and $\underline{\tilde{v}}^{(k)}$ is an error vector, assumed to have a multinormal distribution with mean $\underline{0}$ and precision $\underline{\underline{h}}$, where $\underline{\underline{h}}$ is not known with certainty. Successive $\underline{\tilde{v}}^{(k)}$ are assumed mutually independent. Once $\underline{\beta}^{(k)}$ is fixed, $\underline{x}^{(k)}$ is determined and the actual demand, before it is observed, will satisfy

$$\underline{\tilde{b}}^{(k)} = \underline{\underline{\Pi}}\, \underline{x}^{(k)} + \underline{\tilde{v}}^{(k)} . \tag{84}$$

It may be argued that technological considerations should suffice to determine the matrix $\underline{\underline{A}}$. This, however, may not be possible for a complex industry or, if possible, such an analysis may be more expensive to carry out than the Bayesian analysis proposed here. Moreover, the Bayesian analysis takes forecasting errors explicitly into account, provided they satisfy the assumption of normality. It seems reasonable to ask that the estimated technological coefficients include factors to reflect the uncertainty in the actual demand.

To illustrate the Bayesian analysis, we shall place a Normal-Wishart prior distribution on $(\tilde{\underline{\underline{\Pi}}}, \tilde{\underline{\underline{h}}})$, observe values of $\underline{x}^{(k)}$ and $\underline{\tilde{b}}^{(k)}$ over a period of four years, and combine the sample information with the prior distribution to find the posterior distribution of $(\tilde{\underline{\underline{\Pi}}}, \tilde{\underline{\underline{h}}})$, following the methods of [2].

Let

$$\underline{\tilde{\pi}} = \begin{bmatrix} \tilde{\Pi}_{11} \\ \tilde{\Pi}_{12} \\ \tilde{\Pi}_{21} \\ \tilde{\Pi}_{22} \end{bmatrix} , \quad \underline{\tilde{a}} = \begin{bmatrix} \tilde{a}_{11} \\ \tilde{a}_{12} \\ \tilde{a}_{21} \\ \tilde{a}_{22} \end{bmatrix} , \quad \underline{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} , \quad \underline{p}' = \begin{bmatrix} P'_{11} \\ P'_{12} \\ P'_{21} \\ P'_{22} \end{bmatrix} \tag{85}$$

Writing $\tilde{\underline{\underline{\Pi}}}$ as the vector $\underline{\tilde{\pi}} = \underline{u} - \underline{\tilde{a}}$, if $(\tilde{\underline{\underline{\Pi}}}, \tilde{\underline{\underline{h}}})$ has the Normal-Wishart distribution with parameters $(\underline{\underline{P}}', \underline{\underline{V}}', \underline{\underline{\epsilon}}', \nu')$,

$$f_{NW}^{(2,2)}(\underline{\pi}, \underline{\underline{h}}|\underline{p}', \underline{\underline{V}}', \underline{\underline{\epsilon}}', \nu') = f_N^{(4)}(\underline{\pi}|\underline{p}', \underline{\underline{h}} \otimes \underline{\underline{V}}')\, f_W^{(2)}(\underline{\underline{h}}|\underline{\underline{\epsilon}}', \nu'),$$

then it is shown in [2] that $\tilde{\underline{\pi}}$ has the non-degenerate generalized multivariate Student distribution,

$$f_{SG}^{(4)}(\underline{\pi}|\underline{p}', \underline{\underline{V}}', \underline{\underline{\epsilon}}', \nu') .$$

This distribution is defined in [2]; its mean and variance-covariance matrix were derived in Theorem 3.

Using the results of Theorems 2 and 3 and Corollary 2, we can assign values to the prior parameters as follows. Assume the mean of the marginal prior on $\tilde{\underline{\underline{h}}}^{-1}$ is

$$\bar{\underline{\underline{h}}}^{-1} = \begin{bmatrix} 0.90 & 0.20 \\ 0.20 & 0.30 \end{bmatrix} \tag{86}$$

and that the variance of $\tilde{h}_{22}^{-1}$ in the marginal prior on $\tilde{\underline{\underline{h}}}^{-1}$ is 0.01.

Then, using Theorem 2, the following equations are obtained:

$$\underline{\underline{\epsilon}}' = (\nu'-2) \begin{bmatrix} 0.90 & 0.20 \\ 0.20 & 0.30 \end{bmatrix}$$

$$\frac{2(0.3)^2}{\nu'-4} = 0.01 \tag{87}$$

which yield

$$\underline{\underline{\epsilon}}' = \begin{bmatrix} 18.00 & 4.00 \\ 4.00 & 6.00 \end{bmatrix} \tag{88}$$

$$\nu' = 22 .$$

If the mean of the prior distribution on $\tilde{\underline{\underline{A}}}$ is

$$\bar{\underline{\underline{A}}} = \begin{bmatrix} 0.10 & 1.70 \\ 0.90 & 0.20 \end{bmatrix} \tag{89}$$

then, since $E[\tilde{\underline{\pi}}] = \underline{u} - E[\tilde{\underline{a}}]$, equation (32) yields

$$\underline{p}' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0.10 \\ 1.70 \\ 0.90 \\ 0.20 \end{bmatrix} = \begin{bmatrix} 0.90 \\ -1.70 \\ -0.90 \\ 0.80 \end{bmatrix} . \tag{90}$$

Let

$$\underline{\underline{V}}^{-1} = \underline{\underline{W}} = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}$$

and let the variance-covariance matrix of the prior on $\underline{\tilde{\pi}}$ be denoted $\underline{\underline{U}}$, a (4 x 4) matrix. Then (33) and (87) yield the equation

$$\underline{\underline{U}} = \begin{bmatrix} 0.9w_{11} & 0.9w_{12} & 0.2w_{11} & 0.2w_{12} \\ 0.9w_{21} & 0.9w_{22} & 0.2w_{21} & 0.2w_{22} \\ 0.2w_{11} & 0.2w_{12} & 0.3w_{11} & 0.3w_{12} \\ 0.2w_{21} & 0.2w_{22} & 0.3w_{21} & 0.3w_{22} \end{bmatrix} . \tag{91}$$

It is clear that, in general, a set of (w_{11}, w_{12}, w_{22}) cannot be found which will satisfy (91) exactly. One possible method of procedure is to choose the set (w_{11}, w_{12}, w_{22}) so as to minimize the sum of the squared differences of corresponding elements on the left and right sides of equation (91). Another method is to select three independent equations from the matrix equation (91), solve for the w_{ij}, and, if the variances and covariances of $\underline{\tilde{\pi}}$ implied by this particular set of w_{ij} are sufficiently close to $\underline{\underline{U}}$ use this set. We shall follow this latter method for reasons of computational simplicity. For any particular problem, the terminal cost structure will have bearing upon the best way in which to determine $\underline{\underline{V}}'$.

Here, we assume

$$u_{11} = \mathrm{var}[\tilde{a}_{11}] = 0.0025 = 0.9w_{11}$$

$$u_{44} = \mathrm{var}[\tilde{a}_{22}] = 0.01 \quad = 0.3w_{22}$$

$$u_{14} = \mathrm{cov}[\tilde{a}_{11}, \tilde{a}_{22}] = 0.0007 = 0.2w_{12}$$

which yields

$$\underline{\underline{V}}'^{-1} = \begin{bmatrix} 0.00278 & 0.00350 \\ 0.00350 & 0.03333 \end{bmatrix}$$

and

$$\underline{\underline{V}}' = \begin{bmatrix} 415.0 & -43.6 \\ -43.6 & 34.6 \end{bmatrix} . \tag{92}$$

Suppose now that the following data is collected for the years 1955-1958 (details of the way in which this sample was constructed are presented in Appendix B). The observed industrial output is

$$\underline{x}^{(1)} = \begin{bmatrix} 19.44 \\ 10.98 \end{bmatrix} , \quad \underline{x}^{(2)} \begin{bmatrix} 16.88 \\ 9.63 \end{bmatrix} , \quad \underline{x}^{(3)} = \begin{bmatrix} 23.76 \\ 13.90 \end{bmatrix} , \quad \underline{x}^{(4)} = \begin{bmatrix} 12.72 \\ 6.67 \end{bmatrix} \tag{93}$$

and the observed consumer demand is

$$\underline{b}^{(1)} = \begin{bmatrix} 4.09 \\ 2.99 \end{bmatrix} , \quad \underline{b}^{(2)} = \begin{bmatrix} 4.25 \\ 0.89 \end{bmatrix} , \quad \underline{b}^{(3)} = \begin{bmatrix} 6.49 \\ 3.54 \end{bmatrix} , \quad \underline{b}^{(4)} = \begin{bmatrix} 4.42 \\ 0.09 \end{bmatrix} . \tag{94}$$

This data yields the following sample statistics:

$$\nu=2 \quad , \quad \delta=1 \quad , \quad \Phi=0 \quad , \quad m=2 \tag{95}$$

$$\underline{\underline{B}} = \begin{bmatrix} 4.09 & 4.25 & 6.49 & 4.42 \\ 2.99 & 0.89 & 3.54 & 0.09 \end{bmatrix} \tag{96}$$

$$\underline{\underline{X}} = \begin{bmatrix} 19.44 & 16.88 & 23.76 & 12.72 \\ 10.98 & 9.63 & 13.90 & 6.67 \end{bmatrix} \tag{97}$$

$$\underline{\underline{V}} = \sum_{k=1}^{4} \underline{x}^{(k)} \underline{x}^{(k)t} = \begin{bmatrix} 1389.18 & 791.10 \\ 791.10 & 451.00 \end{bmatrix} \tag{98}$$

$$\underline{\underline{P}}^t = (\underline{\underline{X}}\ \underline{\underline{X}}^t)^{-1}\ \underline{\underline{X}}\ \underline{\underline{B}}^t = \underline{\underline{V}}^{-1}\ \underline{\underline{X}}\ \underline{\underline{B}}^t = \begin{bmatrix} 0.60 & -1.13 \\ -0.98 & 2.01 \end{bmatrix}$$

or

$$\underline{\underline{P}} = \begin{bmatrix} 0.60 & -0.98 \\ -1.13 & 2.01 \end{bmatrix} \tag{99}$$

and

$$\underline{\underline{\epsilon}} = \sum_{k=1}^{4} \underline{\epsilon}^{(k)} \underline{\epsilon}^{(k)t} = \begin{bmatrix} 68.21 & 29.27 \\ 29.27 & 15.84 \end{bmatrix} \tag{100}$$

where

$$\underline{\epsilon}^{(k)} = \underline{b}^{(k)} - \underline{\underline{P}} \, \underline{x}^{(k)} \, .$$

Using the prior-posterior analysis of [2], the following posterior parameters are obtained.

$$\delta''=1 \tag{101}$$

$$\nu'' = \nu'+\nu+m+\delta'+\delta-\delta''-\Phi-1 = 26 \tag{102}$$

$$\underline{\underline{V}}'' = \underline{\underline{V}}'+\underline{\underline{V}} = \begin{bmatrix} 1804.18 & 747.50 \\ 747.50 & 485.60 \end{bmatrix} \tag{103}$$

$$\underline{\underline{P}}'' = [\underline{\underline{P}}'\underline{\underline{V}}' + \underline{\underline{P}} \, \underline{\underline{V}}]\underline{\underline{V}}''^{-1} = \begin{bmatrix} 0.93 & -1.57 \\ -0.78 & 1.37 \end{bmatrix} \tag{104}$$

$$\underline{\underline{\epsilon}}'' = \underline{\underline{\epsilon}}'+\underline{\underline{\epsilon}}+\underline{\underline{P}}'\underline{\underline{V}}'\underline{\underline{P}}'^{t}+\underline{\underline{P}} \, \underline{\underline{V}} \, \underline{\underline{P}}^{t}-\underline{\underline{P}}''\underline{\underline{V}}''\underline{\underline{P}}''^{t} = \begin{bmatrix} 86.87 & 36.40 \\ 36.40 & 33.40 \end{bmatrix} . \tag{105}$$

After the sample (93) and (94) has been drawn, the posterior expected values of $\tilde{\underline{\underline{\Pi}}}$, $\tilde{\underline{\underline{A}}}$, and $\tilde{\underline{\underline{h}}}^{-1}$ are

$$E[\tilde{\underline{\underline{\Pi}}}] = \underline{\underline{P}}'' = \begin{bmatrix} 0.93 & -1.57 \\ -0.78 & 1.37 \end{bmatrix} \tag{106}$$

$$E[\tilde{\underline{\underline{A}}}] = \underline{\underline{I}}-\underline{\underline{P}}'' = \begin{bmatrix} 0.07 & 1.57 \\ 0.78 & -0.37 \end{bmatrix} \tag{107}$$

$$E[\tilde{\underline{\underline{h}}}^{-1}] = \frac{1}{\nu''-2} \underline{\underline{\epsilon}}'' = \begin{bmatrix} 3.62 & 1.52 \\ 1.52 & 1.39 \end{bmatrix} . \tag{108}$$

The actual parameters underlying the sample (93) and (94) are the technological matrix

$$\underline{\underline{A}}^* = \begin{bmatrix} 0.05 & 1.20 \\ 0.40 & 0.10 \end{bmatrix} \tag{109}$$

and the variance-covariance matrix of the noise vector,

$$\underline{\underline{h}}^{*^{-1}} = \begin{bmatrix} 1.20 & 0.08 \\ 0.08 & 0.50 \end{bmatrix} . \tag{110}$$

The prior expected values of $\tilde{\underline{\underline{A}}}$ and $\tilde{\underline{\underline{h}}}^{-1}$ are given by (89) and (86).

The posterior mean of $\tilde{\underline{\underline{A}}}$ poses a problem if used as an estimate, because a negative technological coefficient has no physical meaning. Since the posterior expectation of $\tilde{\underline{\underline{A}}}$ reflects, in part, the forecasting errors, perhaps the negative element of equation (107) is an indication that the forecasting errors in our example are excessively large and would not be encountered in a real economy in the steady state. It should be noted that the classical least squares estimate of $\underline{\underline{A}}$ encounters this same difficulty. This estimate is, using the solution (99) to the normal equations implied by (93) and (94),

$$\hat{\underline{\underline{A}}} = \underline{\underline{I}} - \underline{\underline{P}} = \begin{bmatrix} 0.40 & 0.98 \\ 1.13 & -1.01 \end{bmatrix} . \tag{111}$$

Comparison of equations (86), (108), and (110) shows that the posterior expectation of the variance-covariance matrix of the noise differs significantly from both the prior expectation and the actual variance matrix. This, it is believed, is a reflection of the coupling between the prior parameters on $(\tilde{\underline{\pi}}, \tilde{\underline{\underline{h}}})$ described in [1], p. 8, despite the fact that $\nu'=22$. The posterior variances of the diagonal elements of $\tilde{\underline{\underline{h}}}^{-1}$ are, using equation (17),

$$\mathrm{var}[\tilde{h}^{-1}_{11}] = 1.19$$

$$\mathrm{var}[\tilde{h}^{-1}_{22}] = 0.18$$

whereas the prior variances are

$$\mathrm{var}[\tilde{h}^{-1}_{11}] = 0.09$$

$$\mathrm{var}[\tilde{h}^{-1}_{22}] = 0.01 .$$

If the prior parameter $\nu'=52$ had been used, then

$$\underline{\underline{\epsilon}}' = \begin{bmatrix} 45.00 & 10.00 \\ 10.00 & 15.00 \end{bmatrix} \tag{112}$$

while $\underline{p}'$ and $\underline{\underline{V}}'$ remain as given by equations (90) and (92). The posterior parameters are unchanged except that

$$\nu''=56 \tag{113}$$

and

$$\underline{\underline{\epsilon}}'' = \begin{bmatrix} 113.87 & 42.40 \\ 42.40 & 42.40 \end{bmatrix}, \tag{114}$$

In this case the posterior expectation of $\underline{\underline{\tilde{h}}}^{-1}$ is

$$E[\underline{\underline{\tilde{h}}}^{-1}] = \frac{1}{54}\,\underline{\underline{\epsilon}}'' = \begin{bmatrix} 2.11 & 0.79 \\ 0.79 & 0.79 \end{bmatrix} \tag{115}$$

while the posterior variances of the diagonal elements of $\underline{\underline{\tilde{h}}}^{-1}$ are

$$\mathrm{var}[\tilde{h}_{11}^{-1}] = 0.17$$

$$\mathrm{var}[\tilde{h}_{22}^{-1}] = 0.024 \quad .$$

The coupling effect, while still present, is diminished.

One final point concerning the model of this section is worth mentioning. The determination of the actual consumer demand may pose a problem, since the vector $\underline{b}^{(k)}$ must include demands which were not satisfied, as well as demands which could be satisfied. In fact, shortages in the output $\underline{x}^{(k)}$ will affect demand in that, if shortages are known to exist, some potential demands may never be presented. On the other hand, an excess of demand over that forecast will stimulate further production. These factors are not included in the Leontief static model.

APPENDIX A

MEAN AND VARIANCE OF THE INVERTED GENERALIZED BETA DENSITY FUNCTIONS

1. Non-Standardized Inverted Generalized Beta Distribution

The r x r random matrix $\tilde{\underline{\underline{V}}}$ is said to have the non-standardized inverted generalized Beta distribution with parameters (a, b, $\underline{\underline{C}}$) if $\tilde{\underline{\underline{V}}}$ has the density function*

$$f_{iB}^{(r)}(\underline{\underline{V}}|a, b, \underline{\underline{C}}) = \frac{|\underline{\underline{C}}|^{b}}{B_r(a,b)} \frac{|\underline{\underline{V}}|^{a-\rho}}{|\underline{\underline{V}}+\underline{\underline{C}}|^{a+b}} \tag{A1}$$

where

$\rho = \frac{1}{2}(r+1)$

$a > \frac{1}{2}(r-1)$

$b > \frac{1}{2}(r-1)$

$\underline{\underline{C}}$ is PDS

and the range of $\tilde{\underline{\underline{V}}}$ is $\{\underline{\underline{V}}|\underline{\underline{V}}$ is PDS$\}$. If we define the generalized Gamma function,

$$\Gamma_r(x) = \pi^{r(r-1)/4}\, \Gamma(x)\Gamma(x-\tfrac{1}{2})\ldots\Gamma(x-\tfrac{r-1}{2}) ,$$

where $\Gamma(x)$ is the Gamma function, then

$$B_r(a,b) = \frac{\Gamma_r(a)\Gamma_r(b)}{\Gamma_r(a+b)}$$

for $a > \frac{1}{2}(r-1)$ and $b > \frac{1}{2}(r-1)$.

Theorem A1: If $\tilde{\underline{\underline{V}}}$ is an r x r PDS matrix which has the non-standardized inverted generalized Beta distribution with parameters (a, b, $\underline{\underline{C}}$), then

$$E[\tilde{\underline{\underline{V}}}] = \frac{a}{b-\rho}\underline{\underline{C}} \quad , \qquad b > \rho \tag{A2}$$

and

$$\begin{aligned} \operatorname{cov}[\tilde{V}_{ij},\tilde{V}_{k\ell}] &= \frac{a(a+1)}{3(b-\rho)(b-\rho-1)} (C_{ik}C_{j\ell}+C_{i\ell}C_{jk}) \\ &+ \frac{a[(1-2a)(b-\rho)+3a]}{3(b-\rho)^2(b-\rho-1)} C_{ij}C_{k\ell} \quad , \quad b > \rho + 1 \ . \end{aligned} \tag{A3}$$

If i=k and j=ℓ, (A3) reduces to

---

*[1], p. 10.

$$\text{var}[\tilde{V}_{ij}] = \frac{a(a+1)}{3(b-\rho)(b-\rho-1)} C_{ii}C_{jj} + \frac{a[(2-a)(b-\rho)+3a]}{3(b-\rho)^2(b-\rho-1)} C_{ij}^2, \; b > \rho + 1 \; . \tag{A4}$$

Proof: In paragraph 2.5 of [1] it is shown that, if $(\tilde{\underline{\underline{V}}}, \tilde{\underline{\underline{h}}})$ has the joint density function

$$D(\underline{\underline{V}}, \underline{\underline{h}}|a, b, \underline{\underline{C}}) = f_W^{(r)}(\underline{\underline{V}}|\underline{\underline{h}}, 2a-r+1)\; f_W^{(r)}(\underline{\underline{h}}|\underline{\underline{C}}, 2b-r+1) \; ,$$

then the marginal density function of $\tilde{\underline{\underline{V}}}$ is $f_{iB}^{(r)}(\underline{\underline{V}}|a, b, \underline{\underline{C}})$. Therefore, using Theorems 1 and 2,

$$E[\tilde{\underline{\underline{V}}}] = E_{\underline{\underline{h}}} E_{\underline{\underline{V}}|\underline{\underline{h}}}[\tilde{\underline{\underline{V}}}] = 2a\; E_{\underline{\underline{h}}}[\tilde{\underline{\underline{h}}}^{-1}] = \frac{2a}{2b-(r+1)} \underline{\underline{C}}, \; 2b > r+1 \tag{A5}$$

from which (A2) follows. Moreover, if $2b > r+3$, by Theorem 1, we have

$$E[\tilde{V}_{ij}\tilde{V}_{k\ell}] = E_{\underline{\underline{h}}} E_{\underline{\underline{V}}|\underline{\underline{h}}}[\tilde{V}_{ij}\tilde{V}_{k\ell}] = 2a\; E_{\underline{\underline{h}}}[\tilde{h}^{-1}_{ik}\tilde{h}^{-1}_{j\ell} + \tilde{h}^{-1}_{i\ell}\tilde{h}^{-1}_{jk} + 2a\tilde{h}^{-1}_{ij}\tilde{h}^{-1}_{k\ell}] \; . \tag{A6}$$

Thus, from equations (24) and (26),

$$E[\tilde{V}_{ij}\tilde{V}_{k\ell}] = \frac{a(a+1)}{3(b-\rho)(b-\rho-1)} (C_{ij}C_{k\ell} + C_{ik}C_{j\ell} + C_{i\ell}C_{jk}) \; , \tag{A7}$$

from which (A3) and (A4) are obtained. Q.E.D.

2. Standardized Inverted Generalized Beta Distribution

The standardized inverted generalized Beta density function with parameters (a,b),

$$f_{iB*}^{(r)}(\underline{\underline{X}}|a, b) = \frac{1}{B_r(a,b)} \; \frac{|\underline{\underline{X}}|^{a-\rho}}{|\underline{\underline{I}}+\underline{\underline{X}}|^{a+b}} \; , \tag{A8}$$

where

$a > \frac{1}{2}(r-1)$

$b > \frac{1}{2}(r-1)$

and the range of the r x r random matrix $\tilde{\underline{\underline{X}}}$ is $\{\underline{\underline{X}}|\underline{\underline{X}} \text{ is PDS}\}$, is derived from (A1) by the transformation*

$$\tilde{\underline{\underline{X}}} = \underline{\underline{T}}^{-1}\; \tilde{\underline{\underline{V}}}(\underline{\underline{T}}^{-1})^t \; . \tag{A9}$$

*[1], pp. 10 and 11.

$\underline{\underline{T}}$ is a non-singular upper triangular matrix such that $\underline{\underline{T}}\ \underline{\underline{T}}^t = \underline{\underline{C}}$.

Theorem A2: If the r x r PDS matrix $\tilde{\underline{\underline{X}}}$ has the standardized inverted generalized Beta distribution with parameters (a, b), then

$$E[\tilde{\underline{\underline{X}}}] = \frac{a}{b-\rho} \underline{\underline{I}} \quad , \qquad b > \rho \tag{A10}$$

and, for α, β=1,...,r,

$$\mathrm{var}[\tilde{X}_{\alpha\beta}] = \frac{a(a+1)}{3(b-\rho)(b-\rho-1)} \quad , \quad \alpha \neq \beta \tag{A11}$$

$$\mathrm{var}[\tilde{X}_{\alpha\alpha}] = \frac{a(a+b-\rho)}{(b-\rho)^2(b-\rho-1)} \tag{A12}$$

$$\mathrm{cov}[\tilde{X}_{\alpha\alpha}, \tilde{X}_{\beta\beta}] = \frac{a[(1-2a)(b-\rho)+3a]}{3(b-\rho)^2(b-\rho-1)} \quad , \quad \alpha \neq \beta \ . \tag{A13}$$

All other convariances are zero.

Proof: Equation (A10) is an immediate consequence of (A9) and (A2), since

$$E[\tilde{\underline{\underline{X}}}] = \underline{\underline{T}}^{-1} E[\tilde{\underline{\underline{V}}}](\underline{\underline{T}}^{-1})^t = \frac{a}{b-\rho} \underline{\underline{T}}^{-1} \underline{\underline{C}}(\underline{\underline{T}}^{-1})^t = \frac{a}{b-\rho} \underline{\underline{I}} \ . \tag{A14}$$

By equation (A9), the generic element of $\tilde{\underline{\underline{X}}}$ is

$$\tilde{X}_{\alpha\beta} = \sum_{i,j} T^{-1}_{\alpha i} T^{-1}_{\beta j} \tilde{V}_{ij} \ ,$$

therefore, using (A7),

$$E[\tilde{X}_{\alpha\beta}\tilde{X}_{\gamma\delta}] = \sum_{i,j,k,\ell} T^{-1}_{\alpha i} T^{-1}_{\beta j} T^{-1}_{\gamma k} T^{-1}_{\delta \ell} E[\tilde{V}_{ij}\tilde{V}_{k\ell}]$$

$$= K \sum_{i,j,k,\ell} T^{-1}_{\alpha i} T^{-1}_{\beta j} T^{-1}_{\gamma k} T^{-1}_{\delta \ell} (C_{ij}C_{k\ell}+C_{ik}C_{j\ell}+C_{i\ell}C_{jk}) \tag{A15}$$

where

$$K = \frac{a(a+1)}{3(b-\rho)(b-\rho-1)} \ .$$

But, if $\delta_{\alpha\beta}$ is the Kronecker delta,

$$\sum_{i,j} T^{-1}_{\alpha i} T^{-1}_{\beta j} C_{ij} = \delta_{\alpha\beta}$$

for all α, β=1,...,r. Thus,

$$\begin{aligned} E[\tilde{X}_{\alpha\beta}\tilde{X}_{\gamma\delta}] &= K(\delta_{\alpha\beta}\delta_{\gamma\delta}+\delta_{\alpha\gamma}\delta_{\beta\delta}+\delta_{\alpha\delta}\delta_{\beta\gamma}) = K \ , \quad \begin{matrix} \alpha=\gamma,\ \beta=\delta,\ \alpha\neq\beta \\ \alpha=\beta,\ \gamma=\delta,\ \alpha\neq\gamma \end{matrix} \\ &= 3K \ , \quad \alpha=\beta=\gamma=\delta \\ &= 0 \ , \quad \text{otherwise} \ . \end{aligned} \tag{A16}$$

The case $\alpha=\delta$, $\beta=\gamma$, $\alpha\neq\beta$ is the same as $\alpha=\gamma$, $\beta=\delta$, $\alpha\neq\beta$, since $\underline{\underline{\tilde{X}}}$ is PDS. Therefore,

$$E[\tilde{X}_{\alpha\beta}^{2}] = E[\tilde{X}_{\alpha\alpha}\tilde{X}_{\gamma\gamma}] = \frac{a(a+1)}{3(b-\rho)(b-\rho-1)} \tag{A17}$$

$$E[\tilde{X}_{\alpha\alpha}^{2}] = \frac{a(a+1)}{(b-\rho)(b-\rho-1)} \quad , \tag{A18}$$

from which (A11), (A12), and (A13) follow. Q.E.D.

APPENDIX B

SAMPLE DATA

1. Radar-Tracking Model

Suppose the target is on course 060°, with speed 600 kts. (cf. Figure B1). Then, if $\underline{w}$ is the target velocity, rounding to the nearest integer,

$$\underline{w} = \begin{bmatrix} 600 \cos 30^{\circ} \\ 600 \sin 30^{\circ} \end{bmatrix} = \begin{bmatrix} 520 \\ 300 \end{bmatrix} . \tag{B1}$$

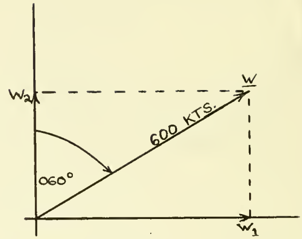

Figure B1

Let the covariance matrix of the error in the radar-derived velocity be

$$\underline{\underline{S}} = \begin{bmatrix} 120 & 70 \\ 70 & 90 \end{bmatrix} , \tag{B2}$$

which implies a standard deviation of 10.95 in the horizontal component of velocity and a standard deviation of 9.49 in the vertical component. The correlation coefficient is 0.673.

Then

$$\underline{\tilde{x}} = \underline{w} + \underline{\tilde{e}} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} \tag{B3}$$

has the bivariate Normal distribution with mean $\underline{w}$ and variance $\underline{\underline{S}}$. Using the results of paragraph 8.2.1 of [3], the marginal distribution of $\tilde{x}_2$ is Normal with mean 300 and variance 90, while the conditional distribution of $\tilde{x}_1$ given $\tilde{x}_2 = x_2$ is Normal with mean

$$m_1 = 520 + \frac{70}{90}(x_2 - 300) = 286.7 + \frac{7}{9} x_2 \tag{B4}$$

and variance

$$\sigma_1^2 = 120 - \frac{(70)(70)}{90} = 65.6 . \tag{B5}$$

With the aid of a table of random numbers uniformly distributed on [0, 1], a sample of 10 Normal (0, 1) random variables, $\tilde{y}^{(i)}$, was obtained, using the probability integral transformation. Using the transformation

$$\tilde{x}_2^{(i)} = 9.487\ \tilde{y}^{(i)} + 300.00 \qquad i=1,\ldots,5, \tag{B6}$$

five sample values of $\tilde{x}_2$ were obtained. Using these sample values of x_2 and the remaining Normal (0, 1) samples, five sample values of $\tilde{x}_1$ were obtained from

$$\tilde{x}_1^{(i)} = 8.100\ \tilde{y}^{(5+i)} + \frac{7}{9}\ x_2^{(i)} + 286.7 \quad i=1,\ldots,5. \tag{B7}$$

This data is shown in Tables B-1, B-2, and B-3.

Sample No.	U(0, 1)	N(0, 1)	Sample No.	U(0, 1)	N(0, 1)
1	.8296	0.95	6	.9122	1.35
2	.1361	-1.10	7	.5812	0.21
3	.5072	0.02	8	.8115	0.88
4	.2533	-0.66	9	.7732	0.75
5	.8443	1.01	10	.2910	-0.55

TABLE B-1

NORMAL (0, 1) SAMPLE

i	$y^{(i)}$	$x_2^{(i)} = 9.487\ y^{(i)} + 300.00$
1	0.95	309.01
2	-1.10	289.56
3	0.02	300.19
4	-0.66	293.74
5	1.01	309.58

TABLE B-2

SAMPLE VALUES OF $\tilde{x}_2$

i	$y^{(5+i)}$	$x_2^{(i)}$	$x_1^{(i)} = 8.1\, y^{(5+i)} + \frac{7}{9} x_2^{(i)} + 286.7$
1	1.35	309.01	539
2	0.21	289.56	514
3	0.88	300.19	528
4	0.75	293.74	521
5	-0.55	309.58	524

TABLE B-3

SAMPLE VALUES OF $\tilde{x}_1$

To simplify the computations, $\tilde{x}_1^{(i)}$ and $\tilde{x}_2^{(i)}$ were rounded to the nearest integer to yield the sample

$$\underline{x}^{(1)} = \begin{bmatrix} 539 \\ 309 \end{bmatrix}, \quad \underline{x}^{(2)} = \begin{bmatrix} 514 \\ 290 \end{bmatrix}, \quad \underline{x}^{(3)} = \begin{bmatrix} 528 \\ 300 \end{bmatrix}$$

$$\underline{x}^{(4)} = \begin{bmatrix} 521 \\ 294 \end{bmatrix}, \quad \underline{x}^{(5)} = \begin{bmatrix} 524 \\ 310 \end{bmatrix}. \tag{B8}$$

This sample has mean

$$\bar{\underline{x}} = \frac{1}{5} \sum_{k=1}^{5} \underline{x}^{(k)} = \begin{bmatrix} 525.2 \\ 300.6 \end{bmatrix} \tag{B9}$$

and unbiassed sample variance

$$\check{\underline{x}} = \frac{1}{4} \sum_{k=1}^{5} [\underline{x}^{(k)} - \bar{\underline{x}}]\, [\underline{x}^{(k)} - \bar{\underline{x}}]^t = \begin{bmatrix} 85.6 & 62.3 \\ 62.3 & 78.8 \end{bmatrix}. \tag{B10}$$

2. Leontief's Static Model

Suppose the technological matrix is

$$\underset{\sim}{A}^* = \begin{bmatrix} 0.05 & 1.20 \\ 0.40 & 0.10 \end{bmatrix} \tag{B11}$$

which implies that

$$\underline{\underline{\Pi}}^* = \underline{\underline{I}}-\underline{\underline{A}}^* = \begin{bmatrix} 0.95 & -1.20 \\ -0.40 & 0.90 \end{bmatrix} . \tag{B12}$$

Assume the forecast demands for 1955-1958 were

$$\underline{\beta}^{(1)} = \begin{bmatrix} 5.3 \\ 2.1 \end{bmatrix} , \quad \underline{\beta}^{(2)} = \begin{bmatrix} 4.5 \\ 1.9 \end{bmatrix} , \quad \underline{\beta}^{(3)} = \begin{bmatrix} 5.9 \\ 3.0 \end{bmatrix} , \quad \underline{\beta}^{(4)} = \begin{bmatrix} 4.1 \\ 0.9 \end{bmatrix} . \tag{B13}$$

Equation (82) then yields, for the outputs $\underline{x}^{(k)}$,

$$\underline{x}^{(k)} = \underline{\underline{\Pi}}^{*-1} \underline{\beta}^{(k)}$$

or

$$\underline{x}^{(1)} = \begin{bmatrix} 19.44 \\ 10.98 \end{bmatrix} , \quad \underline{x}^{(2)} = \begin{bmatrix} 16.88 \\ 9.63 \end{bmatrix} , \quad \underline{x}^{(3)} = \begin{bmatrix} 23.76 \\ 13.90 \end{bmatrix} , \quad \underline{x}^{(4)} = \begin{bmatrix} 12.72 \\ 6.67 \end{bmatrix} . \tag{B14}$$

Let the noise vector,

$$\underline{\tilde{v}} = \begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{bmatrix}$$

have the bivariate normal distribution with mean $\underline{0}$ and variance

$$\underline{\underline{h}}^{*-1} = \begin{bmatrix} 1.20 & 0.08 \\ 0.08 & 0.50 \end{bmatrix} . \tag{B15}$$

A table of random numbers uniformly distributed on [0, 1] then yields the Normal (0, 1) sample values listed in Table B-4.

Sample No.	U(0, 1)	$y_2 \sim N(0, 1)$	Sample No.	U(0, 1)	$y_1 \sim N(0, 1)$
1	0.896	1.26	5	0.207	-1.24
2	0.076	-1.43	6	0.468	-0.08
3	0.780	0.77	7	0.678	0.46
4	0.127	-1.14	8	0.659	0.41

TABLE B-4

NORMAL (0, 1) SAMPLE

The marginal distribution of $\tilde{v}_2$ is Normal with mean 0 and variance 0.50. Thus, if $\tilde{y}_2$ is N(0, 1),

$$\tilde{v}_2 = 0.707\ \tilde{y}_2 \quad . \tag{B16}$$

This transformation is effected in Table B-5, using sample values 1-4 from Table B-4.

i	$y_2^{(i)}$	$v_2^{(i)} = 0.707\ y_2^{(i)}$
1	1.26	0.89
2	-1.43	-1.01
3	0.77	0.54
4	-1.14	-0.81

TABLE B-5

SAMPLE VALUES OF $\tilde{v}_2$

The conditional distribution of $\tilde{v}_1$ gives $\tilde{v}_2 = v_2$ is Normal with mean

$$m_1 = \frac{0.08 v_2}{0.5} = 0.16 v_2 \tag{B17}$$

and variance

$$\sigma_1^2 = 1.20 - \frac{(0.08)(0.08)}{0.5} = 1.1872 \quad . \tag{B18}$$

Then, since $\tilde{y}_1$ is N(0, 1),

$$\tilde{v}_1 = 1.090\ \tilde{y}_1 + 0.16\ v_2 \ . \tag{B19}$$

This transformation is displayed in Table B-6, using the sample values 5-8 from Table B-4 and values of v_2 from Table B-5.

i	$y_1^{(4+i)}$	$v_2^{(i)}$	$v_1^{(i)} = 1.090\ y_1^{(4+i)} + 0.16\ v_2^{(i)}$
1	-1.24	0.89	-1.21
2	-0.08	-1.01	-0.25
3	0.46	0.54	0.59
4	0.41	-0.81	0.32

TABLE B-6

SAMPLE VALUES OF $\tilde{v}_1$

The noise sample is then

$$\underline{v}^{(1)} = \begin{bmatrix} -1.21 \\ 0.89 \end{bmatrix}, \quad \underline{v}^{(2)} = \begin{bmatrix} -0.25 \\ -1.01 \end{bmatrix}, \quad \underline{v}^{(3)} = \begin{bmatrix} 0.59 \\ 0.54 \end{bmatrix}, \quad \underline{v}^{(4)} = \begin{bmatrix} 0.32 \\ -0.81 \end{bmatrix}. \tag{B20}$$

This sample has mean

$$\bar{\underline{v}} = \frac{1}{4} \sum_{k=1}^{4} \underline{v}^{(k)} = \begin{bmatrix} -0.14 \\ -0.10 \end{bmatrix} \tag{B21}$$

and unbiassed sample variance

$$\underline{V} = \frac{1}{3} \sum_{k=1}^{4} [\underline{v}^{(k)} - \bar{\underline{v}}]\,[\underline{v}^{(k)} - \bar{\underline{v}}]^t = \begin{bmatrix} 0.63 & -0.27 \\ -0.27 & 0.91 \end{bmatrix}. \tag{B22}$$

Combining (B20) with the forecasts (B13) to obtain the actual consumer demand,

$$\underline{b}^{(k)} = \underline{\beta}^{(k)} + \underline{v}^{(k)}, \tag{B23}$$

the vectors $\underline{b}^{(k)}$ are

$$\underline{b}^{(1)} = \begin{bmatrix} 4.09 \\ 2.99 \end{bmatrix}, \quad \underline{b}^{(2)} = \begin{bmatrix} 4.25 \\ 0.89 \end{bmatrix}, \quad \underline{b}^{(3)} = \begin{bmatrix} 6.49 \\ 3.54 \end{bmatrix}, \quad \underline{b}^{(4)} = \begin{bmatrix} 4.42 \\ 0.09 \end{bmatrix}. \tag*{B(24)}$$

REFERENCES

[1] Albert Ando and G.M. Kaufman, "Bayesian Analysis of the Independent Multinormal Process--Neither Mean Nor Precision Known, Part I," Working Paper No. 41-63, Alfred P. Sloan School of Management, Massachusetts Institute of Technology, Cambridge (Revised March, 1964).

[2] G.M. Kaufman and Albert Ando, "Bayesian Analysis of the Reduced Form System," Working Paper No. 79-64, Alfred P. Sloan School of Management, Massachusetts Institute of Technology, Cambridge (to be published).

[3] H. Raffia and R. Schlaifer, Applied Statistical Decision Theory, Boston, (1961).

[4] T.W. Anderson, An Introduction to Multivariate Statistical Analysis, New York, (1958).

BASEMENT

Date Due

OCT 16 '75		
SEP 11 '78	[redacted]	
APR 10 '79	DEC 15 1986	
SEP 15 '79	JUL 10 1989	
FEB 07 '80		
JUL 12 '8[illegible]		
[redacted]		
[redacted]81		
[redacted]		
[redacted]		

Lib-26-67

MIT LIBRARIES
3 9080 003 900 260
85-64
MIT LIBRARIES
3 9080 003 869 218
86-64
MIT LIBRARIES
3 9080 003 900 203
87-64
MIT LIBRARIES
3 9080 003 900 211
88-64
MIT LIBRARIES
3 9080 003 900 476
89-64
MIT LIBRARIES
3 9080 003 869 457
90-64
MIT LIBRARIES
3 9080 003 869 507
91-64
MIT LIBRARIES
3 9080 003 900 500
92-64
MIT LIBRARIES
3 9080 003 900 484
93-64
MIT LIBRARIES
3 9080 003 869 465
94-64
MIT LIBRARIES
3 9080 003 900 468
97-64
MIT LIBRARIES
3 9080 003 869 481
98-64
MIT LIBRARIES
3 9080 003 869 531
99-64
95
96

1751891
ean